ACADEMIC PRESS
J. Math. Anal. Appl. 269 (2002) 98–128
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.academicpress.com

Sufficient conditions for relative minima of broken extremals in optimal control theory ☆

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Received 23 April 2001; received in revised form 9 October 2001; accepted 1 November 2001

Submitted by B.S. Mordukhovich

Abstract

We formulate a version of the method of characteristics based on parametrizations of extremals by their terminal values. Sufficient conditions are given for imbedding a reference trajectory into a local field of broken extremals. For a problem with terminal manifold of codimension 1 it is shown that a broken extremal is a relative minimum if (i) the restrictions of the flow to intervals where the control is continuous have nonsingular partial derivatives with respect to the parameter and (ii) the switching surfaces are crossed transversally. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

In this paper we develop sufficient conditions for a relative minimum for broken extremals in an optimal control problem based on the method of characteris-

☆ This material is based upon work supported by the National Science Foundation under grant No. 9971747. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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PII: S0022-247X(02)00008-2

tics. Briefly, we consider the problem to minimize

$$J(u) = \int_{\tau}^{T} L(t, x, u)\, dt + \varphi(T, x(T)) \tag{1.1}$$

subject to dynamics $\dot{x} = f(t, x, u)$, initial condition $x(\tau) = \xi$, and terminal constraint $(T, x(T)) \in N$ where N is an imbedded submanifold of $\mathbb{R} \times \mathbb{R}^n$. The state space is $\mathbb{R}^n$ and admissible controls are Lebesgue-measurable functions u which take values in a control set $U \subset \mathbb{R}^m$.

The Pontryagin Maximum Principle [1] gives first-order necessary conditions for optimality of an input–trajectory pair (x_*, u_*). Pairs which satisfy the necessary conditions of the Maximum Principle are called extremals. Intuitively (a precise definition will be given in Section 2.2), if the corresponding controls have discontinuities we speak of *broken extremals*. We say a reference extremal Γ defines a *relative minimum* if there exists a set $\mathcal{T}$ of trajectories which contains Γ so that Γ minimizes the objective $J(u)$ over all trajectories in $\mathcal{T}$. If $\mathcal{T}$ consists of all trajectories which steer ξ into N, then Γ is optimal.

In general the necessary conditions for optimality of the Maximum Principle are not sufficient and extremals need not be optimal. In fact, in optimal control, like in the classical calculus of variations, there exists a sizable gap between the theories of necessary and sufficient conditions. Sufficiency theories usually center around the related notions of fields of extremals or Hamilton–Jacobi theory. The latter is realized in optimal control by studying solutions to the so-called Hamilton–Jacobi–Bellman equation which for problem (1.1) reads

$$V_t(t, x) + \min_{u \in U} \{ V_x(t, x) f(t, x, u) + L(t, x, u) \} \equiv 0 \tag{1.2}$$

subject to the boundary condition $V(t, x) = \varphi(t, x)$ for $(t, x) \in N$. For systems with special structure, like linear–quadratic optimal control problems, this equation may be solved explicitly. In general, however, even if the equation in (1.2) is strictly convex in u, the minimization leads to a highly nonlinear equation which can no longer be solved explicitly. Quite often, these solutions also exhibit singularities which makes their analysis a difficult and challenging problem.

Various approaches have been pursued in the literature to tackle this problem. Recently PDE techniques seem to be at the forefront of research in the context of viscosity solutions [2,3]. The classical method to solve first-order partial differential equations is the method of characteristics [4], and this method can be adjusted for application to the optimal control problem. Indeed, the characteristic equations for the Hamilton–Jacobi–Bellman equation are precisely the system and adjoint equations given in the Maximum Principle [5] and the minimum condition in the Maximum Principle becomes the minimum condition in (1.2). This is the background of traditional optimal control techniques on *regular synthesis* going back to Boltyansky [6] in the early sixties. By now several refinements of Boltyansky's original construction have been given (e.g., Brunovsky [7] and

Piccoli and Sussmann [8]), yet all of these retain the basic principle of constructing the optimal value function by synthesizing a family of extremals and their corresponding controls which solve (1.2). Newer results differ from the original versions in the much weaker technical assumptions which one needs to make to handle the singularities which arise in the construction of the value function. Up to now the most general version in making minimal assumptions is given in the paper by Piccoli and Sussmann [8].

In this paper we develop the method of characteristics based on injectivity properties of the flow of extremals for families of broken extremals. This flow is obtained by integrating the equations given in the necessary conditions for optimality backward from the terminal manifold. While the method of characteristics is a classical and well-known technique in optimal control, our formulation emphasizing the mapping properties is to the best of our knowledge new. A similar approach was pursued by Young [9] and Nowakowski [10], but in this approach the role of the mapping properties is very much clouded by imprecise definitions like “descriptive” maps. Consequently Young’s results have found little application, although many of the geometric problems in constructing a synthesis are explained and documented in his book. In our view his constructions are obstructed by a strive for too much generality. In contrast, it was not our aim to be of utmost generality, but to keep the results easily verifiable and directly applicable.

The literature on higher-order necessary or sufficient conditions for optimality of bang–bang controls is scarce. One of the earliest papers dealing with this issue explicitly is [11] by Bressan who used the notion of directional convexity to develop high-order tests for optimality of bang–bang controls. Similar constructions were later used by Schättler [12] in the local analysis of the optimality of bang–bang controls in dimension three. Sarychev [13] has introduced extended first and second variations for bang–bang controls which add Dirac measures at the switching times and he obtained corresponding first- and second-order conditions for optimality of bang–bang controls. Agrachev and Gamkrelidze [14,15] initiated a general research program which extends the use of Hamiltonian methods, especially symplectic geometry, in the study of optimal control problems. Early results, giving necessary and sufficient conditions for optimality of extremal trajectories in terms of Morse and Maslov indices, were general and abstract, but rather difficult to apply. These results were developed further by Agrachev, Stefani and Zezza to obtain general sufficient conditions for strong minima of trajectories [16]. While this paper was under review we have become aware of a recent paper by the same authors [17] which develops these conditions for bang–bang controls and also gives an algorithmic way to evaluate the positive definiteness of an associated quadratic form which allows to determine optimality. Essentially these conditions are obtained by varying the switching times of the reference control. The assumptions made in [17] are in the spirit of our results below (except that Agrachev, Stefani and Zezza consider the case with an arbitrary terminal manifold, but for systems which are linear in the control), but the actual computations are different and the precise relations between the two approaches still need to be analyzed. The conditions we develop below are applicable to general systems and reference trajectories which have “corners” (in the sense of Definition 2.3), not just to bang–bang controls. However, we need that the reference extremal can locally be imbedded into a sufficiently smooth field of extremals. While this is certainly not always the case, typically it will be satisfied for a generic trajectory. In fact, the assumptions made in [17] precisely relate to the ability of imbedding a bang–bang extremal into a local field of bang–bang extremals by varying the switching times.

The organization of the paper is as follows: In Section 2 we develop the method of characteristics emphasizing mapping properties of the family of extremals. By postponing injectivity requirements to the last step, the significance of specific assumptions becomes clear. Differentiable solutions to the Hamilton–Jacobi–Bellman equation are constructed for both continuous controls and broken extremals. We show that the value function remains continuously differentiable at switching surfaces which are codimension 1 imbedded submanifolds and are crossed transversally by the optimal flow. These results, some of them classical, form the background for Section 3 where we localize the construction to the case when the parametrization set is a small neighborhood of a reference value which represents a reference trajectory. Conditions are given which allow to imbed the reference trajectory into a local field of extremals and local characterizations of regularity and transversality conditions at switchings are derived. In order to simplify the presentation in this paper we only consider the considerably less technical case when the terminal manifold N has codimension 1. In [18] the proper concepts are developed under the terminology *terminal configuration* which allows for a codimension 1 stratified set off the terminal manifold where the flow can overlap, but is memoryless.

2. The method of characteristics in optimal control

We show how a solution to the Hamilton–Jacobi–Bellman equation can be constructed on a set R from a parametrized family of normal extremals which satisfy the Maximum Principle and cover the set R injectively. The construction itself is classical and is the method of characteristics adapted to the optimal control problem. The constructions below are based on ideas of Knobloch [19] and give a refinement of the proof in [20] by moving the entire construction from the state space into the parameter space. Initially no injectivity assumptions are made in the definitions and the precise roles of assumptions come out. The construction is carried out both for continuous and piecewise continuous controls to cover broken extremals.

2.1. Formulation of the problem

Let U be a subset of $\mathbb{R}^m$, the *control set*, and denote by $\mathcal{U}$ the class of all locally bounded Lebesgue measurable maps defined on some interval $I \subset \mathbb{R}$ with values in U, $u : I \to U$, the *space of* (*admissible*) *controls*. Suppose $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $(t,x,u) \mapsto f(t,x,u)$, *the dynamics of the control system*, is a continuous map which for fixed $t \in \mathbb{R}$ is r-times continuously differentiable in (x,u). Then, given a control $u \in \mathcal{U}$ defined over the interval I, the resulting time-varying *controlled vector field* $f^u(t,x) = f(t,x,u(t))$ satisfies the C^r-Caratheodory conditions [21] and therefore the initial value problem

$$\dot{x} = f^u(t,x) = f(t,x,u(t)), \qquad x(\tau) = \xi, \quad \tau \in I, \tag{2.1}$$

has a unique solution $x^u(t;\tau,\xi)$ defined on an maximal open interval of definition $J = (t_-^u(\tau,\xi), t_+^u(\tau,\xi)) \subset I$. That is, the function x^u is absolutely continuous on J, satisfies the initial condition $x^u(\tau;\tau,\xi) = \xi$, and satisfies Eq. (2.1) almost everywhere on J. Furthermore, as a function of the initial time τ and the initial value ξ, x^u is r-times continuously differentiable [21]. Call this solution x^u the *trajectory* corresponding to the control $u \in \mathcal{U}$ and call the pair (x^u,u) a *controlled trajectory*. Henceforth, for convenience of notation we drop the superscript u.

Let N be a k-dimensional imbedded submanifold of (t,x)-space $\mathbb{R} \times \mathbb{R}^n$, the *terminal manifold*. Thus near every point $q \in N$ there exist an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ containing q and a map $\Psi : \Omega \to \mathbb{R} \times \mathbb{R}^n$ with components ψ_i, $i = 0,\dots,n-k$, which have linearly independent gradients $\nabla\psi_i$ and satisfy $N \cap \Omega = \{(t,x) \in \Omega\colon\ \psi_i(t,x) = 0,\ i = 0,\dots,n-k\}$. The function $\varphi : N \to \mathbb{R}$ is a penalty term which is defined and sufficiently smooth on the terminal manifold N. Locally φ can always be extended to a smooth function in $\mathbb{R} \times \mathbb{R}^n$. Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $(t,x,u) \mapsto L(t,x,u)$, be a continuous function which for t fixed is r-times continuously differentiable in (x,u). This function will define the *Lagrangian* of the optimal control problem. We consider the problem to minimize over $\mathcal{U}$ a cost functional given in Bolza form as

$$\mathcal{J}(u;\tau,\xi) = \int_\tau^T L(t,x,u)\,ds + \varphi(T,x(T)) \tag{2.2}$$

subject to the following dynamics, initial and terminal conditions:

$$\dot{x} = f^u(t,x) = f(t,x,u), \qquad x(\tau) = \xi, \qquad (T,x(T)) \in N. \tag{2.3}$$

The terminal time T is free. (A fixed terminal time would be modelled as additional terminal constraint in this set-up.)

The Pontryagin Maximum Principle [1] gives necessary conditions for a controlled trajectory (x,u) to be optimal. In our notation we distinguish between tangent vectors which we write as column vectors (such as x, $f(t,x,u)$, etc.) and cotangent vectors which we write as row vectors (like the multipliers λ

and ν in the statement of the Maximum Principle below). We denote the space of n-dimensional row vectors by $(\mathbb{R}^n)^*$. Define the Hamiltonian function H, $H:\mathbb{R}\times[0,\infty)\times(\mathbb{R}^n)^*\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, as

$$H(t,\lambda_0,\lambda,x,u)=\lambda_0 L(t,x,u)+\lambda f(t,x,u). \tag{2.4}$$

Theorem 2.1 [1,5,22]. *Suppose the controlled trajectory (x,u) defined over the interval $[\tau,T]$ is optimal. Then there exist a constant $\lambda_0\geqslant 0$, a covector $\nu\in(\mathbb{R}^{n+1-k})^*$ and an absolutely continuous function $\lambda:[\tau,T]\to(\mathbb{R}^n)^*$ which is a solution to the adjoint equation*

$$\dot\lambda(t)=-\lambda_0 L_x(t,x(t),u(t))-\lambda(t)f_x(t,x(t),u(t)), \tag{2.5}$$

such that $(\lambda_0,\lambda(t))\neq 0$ for all $t\in[\tau,T]$ and the following minimum condition is satisfied almost everywhere in $[\tau,T]$:

$$H(t,\lambda_0,\lambda(t),x(t),u(t))=\min_{v\in U}H(t,\lambda_0,\lambda(t),x(t),v). \tag{2.6}$$

In addition the vector $(H+\lambda_0\varphi_t,-\lambda+\lambda_0\varphi_x)$ is orthogonal to the terminal constraint at the endpoint; *i.e., at the terminal time T we have the following transversality condition*:

$$0=\lambda_0\varphi_t+\nu D_t\Psi+H,\qquad \lambda=\lambda_0\varphi_x+\nu D_x\Psi. \tag{2.7}$$

We call controlled trajectories (x,u) for which there exist multipliers λ_0, λ and ν such that the conditions of the Maximum Principle are satisfied *extremals* and sometimes we refer to the triple (λ,x,u) as an *extremal lift*. Note that the conditions are linear in the multipliers (λ_0,λ) and thus it is possible to normalize this vector. In particular, if $\lambda_0>0$, then we can divide by λ_0 and thus assume $\lambda_0=1$. These kind of extremals are called *normal* while extremals with $\lambda_0=0$ are called *abnormal*. The existence of optimal abnormal extremals can in general not be ruled out.

2.2. Parametrized families of extremals

We parametrize extremals through their endpoints in the terminal manifold N (k-dimensional) and the vector ν in the transversality condition which gives the terminal condition for the multiplier λ ($(n+1-k)$-dimensional). However, we also need to enforce the transversality condition (2.7) on H which pins down the terminal time. Hence the parameter space is n-dimensional.

Definition 2.2. A *C^r-parametrized family $\mathcal{E}$ of extremal lifts* is an 8-tuple $(P;\mathcal{T};\xi,\nu;x,u,\lambda_0,\lambda)$ consisting of:

- An n-dimensional manifold P and a pair $\mathcal{T}=(t_{\text{in}},t_f)$ of r-times continuously differentiable functions $t_{\text{in}}:P\to\mathbb{R}$ and $t_f:P\to\mathbb{R}$ which satisfy

$t_{\mathrm{in}}(p) < t_f(p)$ for all $p \in P$. They define the domain of the parametrization as $D = \{(t, p)\colon\ p \in P, t \in I_p = [t_{\mathrm{in}}(p), t_f(p)]\}$. The functions t_{in} and t_f define the (compact) intervals of definition for the controlled trajectories with t_f denoting the terminal time.

- An r-times continuously differentiable function $\xi : P \to N$ which parametrizes the terminal conditions for the states.
- Extremal lifts consisting of controlled trajectories $(x, u) : D \to \mathbb{R}^n \times U$, corresponding adjoint vectors $\lambda_0 : P \to [0, \infty)$ and $\lambda : D \to (\mathbb{R}^n)^*$ and a function $\nu : P \to (\mathbb{R}^{n+1-k})^*$ which parametrizes the terminal conditions of the costates. Specifically, we assume
  (1) the multipliers $(\lambda_0(p), \lambda(t, p))$ are nontrivial for all $t \in I_p$,
  (2) the controls $u = u(\cdot, p)$, $p \in P$, parametrize admissible controls which are continuous in (t, p) and for t fixed depend r-times continuously differentiable on p with the derivatives continuous in (t, p),
  (3) the trajectories $x = x(t, p)$ solve the following terminal value problems for the dynamics
  $$\dot{x}(t, p) = f(t, x(t, p), u(t, p)), \qquad x(t_f(p), p) = \xi(p) \in N,$$
  (4) the costate $\lambda = \lambda(t, p)$ solves the corresponding adjoint equation
  $$\begin{aligned}\dot{\lambda}(t, p) = {} & -\lambda_0(p) L_x(t, x(t, p), u(t, p)) \\ & - \lambda(t, p) f_x(t, x(t, p), u(t, p)),\end{aligned}$$
  with terminal conditions
  $$\lambda(t_f(p), p) = \lambda_0(p)\varphi_x(t_f(p), \xi(p)) + \nu(p) D_x \Psi(t_f(p), \xi(p)),$$
  (5) the controls solve the minimization problem
  $$\begin{aligned}& H(t, \lambda_0(p), \lambda(t, p), x(t, p), u(t, p)) \\ & \quad = \min_{v \in U} H(t, \lambda_0(p), \lambda(t, p), x(t, p), v),\end{aligned}$$
  (6) the transversality condition on the terminal time,
  $$\begin{aligned}& H\big(t_f(p), \lambda_0(p), \lambda(t_f(p), p), \xi(p), u(t_f(p), p)\big) \\ & \quad + \lambda_0(p)\varphi_t(t_f(p), \xi(p)) + \nu(p) D_t \Psi(t_f(p), \xi(p)) = 0,\end{aligned}$$
  holds.

Note that it is not assumed that the parametrization $\mathcal{E}$ of extremals covers the state-space injectively. It follows from standard results about differentiable dependence on parameters of solutions to ordinary differential equations that the trajectories $x = x(t, p)$ and their time-derivatives $\dot{x}(t, p)$ are r-times continuously differentiable in p for fixed t and that the derivatives are continuous jointly in (t, p). These partial derivatives can be calculated as solutions to the corresponding variational equations. Also, at the moment we do not impose any regularity properties on the parameters λ_0 and ν. Consequently the multiplier $\lambda = \lambda(t, p)$ may not be differentiable in p. For the general theory this will not be needed. However, if λ_0 and ν are also r-times continuously differentiable, then the costate $\lambda(t, p)$ has smoothness properties identical to $x(t, p)$. We call this then a *nicely* C^r-parametrized family of extremals.

The definition below gives the necessary modifications if parametrized families of broken extremals are considered.

Definition 2.3. A *C^r-parametrized family of broken extremal lifts* is an 8-tuple $(P; \mathcal{T}; \xi, \nu; x, u, \lambda_0, \lambda)$ where $\mathcal{T} = \{t_{\text{in}} = t_{m+1}, \dots, t_1, t_0 = t_f\}$ is a finite family of r-times continuously differentiable functions $t_j : P \to \mathbb{R}$ which satisfy $t_{j+1}(p) < t_j(p)$ for all $p \in P$, $j = 0, \dots, m$, and the conditions of Definition 2.2 are satisfied piecewise in the sense of the following modifications:

- Let $D_j = \{(t, p)\colon\ p \in P,\ t_{j+1}(p) \leqslant t \leqslant t_j(p)\}$, and $D_j^* = \{(t, p)\colon\ p \in P,\ t_{j+1}(p) < t < t_j(p)\}$. The control $u(\cdot)$ is given by an admissible control $u_j = u_j(t, p)$ on D_j^* which is r-times continuously differentiable in p for t fixed, the partial derivatives in p are continuous on D_j^* and the function u_j itself has a continuous extension to D_j.
- The conditions defining x and λ stay in effect (i.e., $x(\cdot, p)$ and $\lambda(\cdot, p)$ satisfy the system and adjoint equations of the Maximum Principle with $u(\cdot, p)$).

Although these definitions are formulated in general, we very much think of P as a sufficiently small neighborhood of some parameter p_0 corresponding to a reference extremal lift $(x(\cdot, p_0), u(\cdot, p_0), \lambda_0(p_0), \lambda(\cdot, p_0))$. Clearly, and this is seen by looking at simple text-book examples, like, for instance, Bushaw's problem (the problem of steering the double-integrator to the origin in $\mathbb{R}^2$ time-optimally), the smoothness properties required in p in these definitions will seldom be satisfied globally. Locally, however, like in Bushaw's problem, such a structure is typically valid for all but few exceptional trajectories (two abnormal extremals in Bushaw's problem) and simply is a consequence of smooth dependence of solutions of ODE's on parameters and initial conditions.

In order to make this point more explicitly, we briefly describe two types of optimal control problems and outline how C^r-parametrized families of broken extremal lifts naturally arise in this context. Consider the problem Σ_r, $r = 1, 2$, to minimize an objective of the form

$$\int_0^T \left(L(t, x) + \frac{1}{r} u^r \right) dt + \phi(x(T)) \tag{2.8}$$

over all Lebesgue-measurable functions $u : [0, T] \to [0, 1]$ subject to

$$\dot{x} = f(t, x) + u g(t, x), \qquad x(0) = x_0, \tag{2.9}$$

with fixed terminal time T. In the dynamics f and g are time-varying vector fields which satisfy so-called C^{r+1}-Caratheodory conditions [21] which guarantee that for any control u the initial value problem (2.9) has a unique solution which is C^{r+1} in the initial data. Problems of this type are quite common as mathematical models in biomedical problems like the chemotherapy of cancer [23,24] and HIV [25] when the number of cancer cells or infected $CD4^{+}T$ cells needs to be minimized over a prescribed fixed therapy horizon.

It is easy to see that these problems are normal and thus we can normalize $\lambda_0 \equiv 1$ for all extremals. For the quadratic objective the Hamiltonian H is strictly convex in u and over $\mathbb{R}$ has a unique minimum at $u = -\lambda g(t,x)$. Therefore, depending on whether this value lies in the interval $[0,1]$ or not, the minimizing control over $[0,1]$ is given by

$$u = \begin{cases} 0 & \text{if } \lambda g(t,x) \geqslant 0, \\ -\lambda g(t,x) & \text{if } 0 \geqslant \lambda g(t,x) \geqslant -1, \\ 1 & \text{if } \lambda g(t,x) \leqslant -1. \end{cases} \tag{2.10}$$

A nicely C^r-parametrized family of broken extremal lifts is constructed by integrating the system and adjoint equations backward from the terminal time T with terminal conditions

$$x(T,p) = p, \qquad \lambda(T,p) = \frac{\partial \phi}{\partial x}(p); \tag{2.11}$$

i.e., the terminal value of the trajectory is taken as parameter. It follows from standard results on smooth dependence of solutions to ODE on initial data [21] that the solutions $x(t,p)$ and $\lambda(t,p)$ of this system with either one of $u(t,p) \equiv 1$, $u(t,p) = -\lambda g(t,x(t,p))$ or $u(t,p) \equiv 0$ depend smoothly on the terminal condition and thus initially generate a nicely C^r-parametrized family of extremal lifts as defined above. Once switchings between these candidates arise, these smoothness properties will be inherited by the new flow except in degenerate cases when it is not possible to solve the equations

$$\Phi_0(t,p) = \lambda(t,p) g(t,x(t,p)) \equiv 0 \tag{2.12}$$

or

$$\Phi_1(t,p) = \lambda(t,p) g(t,x(t,p)) \equiv -1 \tag{2.13}$$

smoothly for $t = \tau(p)$. By the implicit function theorem this can be done if the time-derivative $\dot{\Phi}_i(t,p)$, $i = 0,1$, does not vanish at the switchings. These time derivatives are easily calculated as

$$\begin{aligned} \dot{\Phi}_i(t,p) = {} & -L_x(t,x(t,p)) g(t,x(t,p)) \\ & + \lambda(t,p)\big\{[f,g](t,x(t,p)) + g_t(t,x(t,p))\big\}, \end{aligned} \tag{2.14}$$

where $[f,g]$ denotes the Lie bracket of the time-varying vector fields $f(t,\cdot)$ and $g(t,\cdot)$. In particular, no information other than the states and costates at the switching are required. Thus, "typically" it is possible to construct a nicely C^r-parametrized family of broken extremal lifts around a reference extremal if the control only switches finitely many times between these three types of candidates for optimality.

Note that in this case the controls actually remain continuous as they "switch" from the interior value $u = -\lambda g(t, x)$ to one of the boundary values. However, the parametrizations in p will no longer be C^r across the "switching surface". In this sense the natural candidates for optimality in the problem Σ_2 are precisely families of broken extremals in the sense of Definition 2.3. The "switches" in this case are seen in higher derivatives of the trajectories at the "switching times" and trajectories of this type naturally fall into the category of *broken extremals* considered in our paper.

Bang–bang controls typically arise as candidates for optimality for the problem Σ_1 when an L_1-type objective is used. Here the controls $u = u(t, p)$ are determined by the zeroes of the switching function

$$\Phi(t, p) = 1 + \lambda(t, p)g(t, x(t, p)). \tag{2.15}$$

Clearly the flows for the constant controls are smooth and again a nicely C^r-parametrized family of broken extremal lifts can be constructed if the equation $\Phi(t, p) \equiv 0$ can be solved smoothly for $\tau = t(p)$ near the switching times. This again will be the case if the derivatives $\dot{\Phi}$ of the switching function do not vanish at the switching times for the reference trajectory. Details of applications of these standard arguments for general systems are given in [18] and also in [23] where some of the results developed here are applied to study locally optimal controls for a mathematical model for cancer chemotherapy.

2.3. The Shadow-Price Lemma

Define the parametrized cost $C : D \to \mathbb{R}$ along a C^r-parametrized family $\mathcal{E}$ of extremal lifts as

$$C(t, p) = \int_t^{t_f(p)} L(s, x(s, p), u(s, p))\, ds + \varphi(t_f(p), \xi(p)). \tag{2.16}$$

It follows from our assumptions and the above smoothness properties that C is continuously differentiable in t on $D^* = \{(t, p)\colon\ t_{\text{in}}(p) < t < t_f(p),\ p \in P\}$ and that both C and its time-derivative $(\partial C/\partial t)(t, p)$ are r-times continuously differentiable in p. The following relation is crucial to the whole construction:

Lemma 2.4 (Shadow-Price Lemma). *Let $\mathcal{E}$ be a C^1-parametrized family of extremal lifts. Then we have that*

$$\lambda_0(p)\frac{\partial C}{\partial p}(t, p) = \lambda(t, p)\frac{\partial x}{\partial p}(t, p). \tag{2.17}$$

Proof. It suffices to show that for p fixed both sides have the same t-derivative and identical values at the terminal time $t_f(p)$. By extending the controls $u = u(t,p)$ continuously beyond the terminal time $t_f(p)$, without loss of generality we may assume that the extremals and covectors are defined on open time-intervals which contain $t_f(p)$ and both sides of Eq. (2.17) are continuously differentiable functions in t for fixed p. We start by calculating the values at the terminal time. For simplicity of notation we drop the arguments. All functions and all their derivatives are evaluated at their proper arguments. We have

$$\begin{aligned}\frac{\partial C}{\partial p}(t_f(p),p) &= \varphi_t\frac{\partial t_f}{\partial p} + \varphi_x\left(f\frac{\partial t_f}{\partial p} + \frac{\partial x}{\partial p}\right) + L\frac{\partial t_f}{\partial p}\\ &= (\varphi_t + \varphi_x f + L)\frac{\partial t_f}{\partial p} + \varphi_x\frac{\partial x}{\partial p}.\end{aligned}$$

Without loss of generality we have assumed that φ is defined and differentiable in the ambient state space. By construction we have $\Psi(t_f(p), x(t_f(p),p)) \equiv 0$ and thus

$$\Psi_t\frac{\partial t_f}{\partial p} + \Psi_x\left(f\frac{\partial t_f}{\partial p} + \frac{\partial x}{\partial p}\right) \equiv 0.$$

Hence, adjoining this equation with multiplier $\nu = \nu(p)$ to $\lambda_0(\partial C/\partial p)$ we get at the endpoint

$$\begin{aligned}\lambda_0\frac{\partial C}{\partial p}(t_f(p),p) &= (\lambda_0\varphi_t + \nu\Psi_t + (\lambda_0\varphi_x + \nu\Psi_x)f + \lambda_0 L)\frac{\partial t_f}{\partial p}\\ &\quad + (\lambda_0\varphi_x + \nu\Psi_x)\frac{\partial x}{\partial p}.\end{aligned}$$

But it follows from the transversality conditions (2.7) that $\lambda = \lambda_0\varphi_x + \nu\Psi_x$ and $0 = \lambda_0\varphi_t + \nu\Psi_t + H$. Thus we obtain at time $t_f(p)$ that

$$\lambda_0\frac{\partial C}{\partial p} = (\lambda_0\varphi_t + \nu\Psi_t + H)\frac{\partial t_f}{\partial p} + \lambda\frac{\partial x}{\partial p} = \lambda\frac{\partial x}{\partial p}. \tag{2.18}$$

It remains to show that both sides have the same time-derivatives. Using the adjoint equation and the variational equation for $\partial x/\partial p$ we get

$$\begin{aligned}\frac{d}{dt}\left\{\lambda(t,p)\frac{\partial x}{\partial p}(t,p)\right\} &= \dot\lambda(t,p)\frac{\partial x}{\partial p}(t,p) + \lambda(t,p)\frac{\partial^2 x}{\partial t\partial p}(t,p)\\ &= (-\lambda_0 L_x - \lambda f_x)\frac{\partial x}{\partial p} + \lambda\left(f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}\right)\\ &= -\lambda_0 L_x\frac{\partial x}{\partial p} - \lambda_0 L_u\frac{\partial u}{\partial p} + H_u\frac{\partial u}{\partial p}\\ &= \lambda_0\frac{\partial^2 C}{\partial t\partial p}(t,p) + H_u\frac{\partial u}{\partial p}.\end{aligned}$$

Hence the proof will be completed by verifying that

$$H_u(t, \lambda_0(p), \lambda(t, p), x(t, p), u(t, p)) \frac{\partial u}{\partial p}(t, p) \equiv 0 \quad \text{on } D. \tag{2.19}$$

To see this, fix a point (t, p) in D^*. For q sufficiently close to p, the control value $v = u(t, q)$ is admissible and therefore it follows from the minimization property of the extremal control $u(t, p)$ that

$$h(\xi) = \lambda_0(p) L(t, x(t, p), u(t, \xi)) + \lambda(t, p) f(t, x(t, p), u(t, \xi))$$

has a local minimum at $\xi = p$. Since this function is differentiable in ξ, we have that

$$0 = \operatorname{grad} h(p) = H_u(t, \lambda_0(p), \lambda(t, p), x(t, p), u(t, p)) \frac{\partial u}{\partial p}(t, p).$$

Hence

$$\frac{d}{dt}\left\{\lambda(t, p) \frac{\partial x}{\partial p}(t, p)\right\} = \lambda_0(p) \frac{\partial^2 C}{\partial t \partial p}(t, p).$$

This proves Eq. (2.17). □

No smoothness is required on the multipliers $\lambda_0(p)$ or $\nu(p)$ for the Shadow-Price Lemma to be valid. We now generalize the Shadow-Price Lemma to families of broken extremals.

Lemma 2.5. *For a C^r-parametrized family of broken extremal lifts the Shadow-Price identity*

$$\lambda_0(p) \frac{\partial C}{\partial p}(t, p) = \lambda(t, p) \frac{\partial x}{\partial p}(t, p) \tag{2.20}$$

holds on each open domain D_j^*, $j = 0, 1, \ldots, m$.

Proof. Without loss of generality we only consider the case $m = 1$. Let

$$D_1^* = \{(t, p)\colon\ t_{\text{in}}(p) < t < t_1(p),\ p \in P\},$$
$$\mathcal{T}_1 = \{(t, p)\colon\ t = t_1(p),\ p \in P\},$$
$$D_0^* = \{(t, p)\colon\ t_1(p) < t < t_f(p),\ p \in P\}.$$

Recall that the parametrized cost $C_0 : D_0 \to \mathbb{R}$ is defined by

$$C_0(t, p) = \int_t^{t_f(p)} L(s, x(s, p), u(s, p))\, ds + \varphi\big(t_f(p), x(t_f(p), p)\big)$$

and let

$$C_1(t,p) = \int\limits_t^{t_1(p)} L(s, x(s,p), u(s,p))\,ds + C_0\big(t_1(p), x(t_1(p),p)\big)$$

denote the parametrized cost on D_1. Also denote by x_i the restriction of x to D_i, $i = 0, 1$. By Lemma 2.4 the result holds on D_0^*. We therefore need to extend the Shadow-Price identity beyond the switching surface to D_1^*. We know that C_1 agrees with C_0 on $\mathcal{T}_1$ and it follows as in the proof of Lemma 2.4 that for $(t,p) \in D_1^*$,

$$\lambda_0(p)\frac{\partial^2 C_1}{\partial t\,\partial p}(t,p) = \frac{d}{dt}\left\{\lambda(t,p)\frac{\partial x_1}{\partial p}(t,p)\right\}. \tag{2.21}$$

However, we need to analyze the behavior of the partial derivatives on the switching surface $\mathcal{T}_1$. Since u_0 and u_1 extend continuously to $\mathcal{T}_1$, we can extend x_0, x_1 and the restriction of λ to D_0 and D_1 as solutions of the corresponding differential equation (not necessarily as extremals) onto a neighborhood of $\mathcal{T}_1$. The following elementary lemma relates the partial derivatives on the switching surface.

Lemma 2.6. *Let $z_0 \in \mathbb{R}^n$ and let Z be an open neighborhood of z_0. Suppose $g: Z \to \mathbb{R}$ and $h: Z \to \mathbb{R}$ are continuously differentiable functions which satisfy $h(z) = 0$ on $\{z \in Z\colon\ g(z) = 0\}$. If $g(z_0) = 0$ and $\nabla g(z_0) \neq 0$, then there exist a neighborhood W of z_0 contained in Z and a continuous function $k: W \to \mathbb{R}$ so that $h(z) = k(z)g(z)$ for $z \in W$ and $\nabla h(z_0) = k(z_0)\nabla g(z_0)$.*

Since we have for $(t,p) \in \mathcal{T}_1 = \{(t,p)\colon\ t - t_1(p) = 0,\ p \in P\}$ that $C_0(t,p) = C_1(t,p)$, and $x_0(t,p) = x_1(t,p) = x(t,p)$, it follows that there exists a continuous vector-valued function $k = (k_0, k_1, \ldots, k_n)$ defined near a reference parameter p_0 such that for $t = t_1(p)$ we have

$$\operatorname{grad} C_1(t,p) = \operatorname{grad} C_0(t,p) + k_0(p)\left(1, -\frac{\partial t_1}{\partial p}(p)\right), \tag{2.22}$$

and for each component $x^{(i)}$ of x_0 and x_1

$$\operatorname{grad} x_1^{(i)}(t,p) = \operatorname{grad} x_0^{(i)}(t,p) + k_i(p)\left(1, -\frac{\partial t_1}{\partial p}(p)\right). \tag{2.23}$$

But from the definition of the parametrized costs we have

$$\frac{\partial C_i}{\partial t}(t,p) = -L(t, x_i(t,p), u_i(t,p)), \quad i = 0, 1, \tag{2.24}$$

and thus

$$\begin{aligned} k_0(p) = {} & L\big(t_1(p), x_0(t_1(p),p), u_0(t_1(p),p)\big) \\ & - L\big(t_1(p), x_1(t_1(p),p), u_1(t_1(p),p)\big). \end{aligned}$$

Similarly, if we set $k(p) = (k_1(p), \dots, k_n(p))$, then

$$\begin{aligned} k(p) &= \dot{x}_1(t_1(p), p) - \dot{x}_0(t_1(p), p) \\ &= f\big(t_1(p), x_1(t_1(p), p), u_1(t_1(p), p)\big) \\ &\quad - f\big(t_1(p), x_0(t_1(p), p), u_0(t_1(p), p)\big). \end{aligned}$$

It follows from the minimum condition (2.6) that the function

$$t \mapsto H(t, \lambda_0(p), \lambda(t, p), x(t, p), u(t, p))$$

is continuous at $t = t_1(p)$, i.e., has the same value for both $u_0(t_1(p), p)$ and $u_1(t_1(p), p)$. Therefore

$$\begin{aligned} &\lambda_0(p)k_0(p) - \lambda(t_1(p), p)k(p) \\ &\quad = \lambda_0 L\big(\dots, u_0(t_1(p), p)\big) + \lambda f\big(\dots, u_0(t_1(p), p)\big) \\ &\qquad - \lambda_0 L\big(\dots, u_1(t_1(p), p)\big) - \lambda f\big(\dots, u_1(t_1(p), p)\big) \\ &\quad = H\big(t_1(p), \dots, u_0(t_1(p), p)\big) - H\big(t_1(p), \dots, u_1(t_1(p), p)\big) \equiv 0, \end{aligned}$$

where some arguments have been suppressed to simplify notation. Hence, from (2.22) and using the Shadow-Price Lemma for C_0, we have that

$$\begin{aligned} \lambda_0(p)\frac{\partial C_1}{\partial p}(t_1(p), p) &= \lambda_0(p)\frac{\partial C_0}{\partial p}(t_1(p), p) - \lambda_0(p)k_0(p)\frac{\partial t_1}{\partial p}(p) \\ &= \lambda(t_1(p), p)\left(\frac{\partial x_0}{\partial p}(t_1(p), p) - k(p)\frac{\partial t_1}{\partial p}(p)\right) \\ &= \lambda(t_1(p), p)\frac{\partial x_1}{\partial p}(t_1(p), p). \end{aligned}$$

Thus

$$\lambda_0(p)\frac{\partial C_1}{\partial p}(t_1(p), p) = \lambda(t, p)\frac{\partial x_1}{\partial p}(t_1(p), p) \quad \text{for all } (t, p) \in D_1^*. \tag{2.25}$$

□

Although the partial derivatives of C and x with respect to the parameters are not continuous at the switching surface, their jumps cancel in the Shadow-Price Lemma. It is this identity which generates solutions to the Hamilton–Jacobi–Bellman equation.

2.4. C^1-solutions to the Hamilton–Jacobi–Bellman equation

We formulate results about the differentiability of the value function for a parametrized family of extremal lifts. We first define feedback controls. Since we need to deal with discontinuous feedbacks, rather than getting into the subtleties of when solutions to ODE's with discontinuous right-hand sides exist, we simply postulate this as definition. We call a function $u^*: R \to U$ an *admissible feedback control* if for each $(\tau, \xi) \in R$ the differential equation $\dot{x} = f(t, x, u^*(t, x))$, $x(\tau) = \xi$, has a unique solution $x^*(t) = x^*(t; \tau, \xi)$ forward in time ($t \geqslant \tau$) and the associated open-loop control $u(t) = u^*(t, x^*(t))$ lies in $\mathcal{U}$. In this case the open-loop and the closed-loop control give rise to the same controlled trajectory from (τ, ξ). The feedback is called optimal on R if each of these open-loop controls is optimal for the problem with initial conditions $(\tau, \xi) \in R$. In this case the value-function is given by $V(\tau, \xi) = \mathcal{J}(u^*; \tau, \xi)$.

Theorem 2.7. *Let $\mathcal{E}$ be a C^r-parametrized family of normal extremal lifts, $r \geqslant 1$, and denote by σ the flow of the trajectories*

$$\sigma: D^* \to \mathbb{R} \times \mathbb{R}^n, \quad (t, p) \mapsto \sigma(t, p) = (t, x(t, p)). \tag{2.26}$$

Suppose the restriction of σ to some open set O is a $C^{1,r}$-diffeomorphism onto an open subset $R \subset \mathbb{R} \times \mathbb{R}^n$ of the state-space, i.e., is injective with a nonvanishing Jacobian determinant on O, is continuously differentiable in t and r-times continuously differentiable in p. Then the function

$$V: R \to \mathbb{R}, \quad V = C \circ \sigma^{-1}, \tag{2.27}$$

is continuously differentiable in t and r-times continuously differentiable in x on R. The function

$$u^*: R \to \mathbb{R}, \quad u^* = u \circ \sigma^{-1}, \tag{2.28}$$

is an admissible feedback control which is continuous in t and r-times continuously differentiable in x. Together the pair (V, u^*) solves the Hamilton–Jacobi–Bellman equation

$$V_t(t, x) + \min_{u \in U}\{V_x(t, x) f(t, x, u) + L(t, x, u)\} \equiv 0 \tag{2.29}$$

on R. Furthermore, the following identities hold in the parameter space on O:

$$V_t(t, x(t, p)) = -H(t, \lambda(t, p), x(t, p), u(t, p)), \tag{2.30}$$

$$V_x(t, x(t, p)) = \lambda(t, p). \tag{2.31}$$

If $\mathcal{E}$ is a nicely C^r-parametrized family of extremal lifts, then V is $(r+1)$-times continuously differentiable in x on R and we also have

$$V_{xx}(t, x(t, p)) = \frac{\partial \lambda^T}{\partial p}(t, p)\left(\frac{\partial x}{\partial p}(t, p)\right)^{-1}. \tag{2.32}$$

Proof. For t fixed, the map $\sigma(t, \cdot): p \mapsto \sigma(t, p)$ is a C^r-diffeomorphism and its inverse $\sigma^{-1}(t, \cdot): x \mapsto \sigma^{-1}(t, x)$ is r-times continuously differentiable in x. Therefore V and u^* are well-defined and r-times continuously differentiable in x.

The smoothness properties in t carry over from the parametrization. Also, the solution to

$$\dot{x} = f(t, x, u^*(t, x)), \qquad x(t_{\text{in}}) = x_0 = x(t_{\text{in}}, p_0) \tag{2.33}$$

is given by $x(\cdot, p_0)$ and thus $u^*(\cdot, x(\cdot, p_0)) = u(\cdot, p_0) \in \mathcal{U}$. Hence u^* is an admissible feedback control. Since $C = V \circ \sigma$, we have that

$$\frac{\partial C}{\partial p}(t, p) = V_x(t, x(t, p))\frac{\partial x}{\partial p}(t, p)$$

and thus, in view of Lemma 2.4 and the fact that $\partial x/\partial p$ is nonsingular, Eq. (2.31) follows. Furthermore,

$$\begin{aligned} -L(t, x(t, p), u(t, p)) &= \frac{\partial C}{\partial t}(t, p) \\ &= V_t(t, x(t, p)) + V_x(t, x(t, p))\dot{x}(t, p) \\ &= V_t(t, x(t, p)) + \lambda(t, p) f(t, x(t, p), u(t, p)) \end{aligned}$$

which gives (2.30). But then the minimum condition in the definition of extremals implies that the pair (V, u^*) solves the Hamilton–Jacobi–Bellman equation: we have for $(t, x) = (t, x(t, p)) \in R$ and arbitrary $v \in U$ that

$$\begin{aligned} &V_t(t, x) + V_x(t, x) f(t, x, v) + L(t, x, v) \\ &\quad = V_t(t, x(t, p)) + V_x(t, x(t, p)) f(t, x(t, p), v) + L(t, x(t, p), v) \\ &\quad \geqslant V_t(t, x(t, p)) + V_x(t, x(t, p)) f(t, x(t, p), u(t, p)) \\ &\qquad + L(t, x(t, p), u(t, p)) \\ &\quad \equiv 0. \end{aligned}$$

If $\mathcal{E}$ is nicely C^r-parametrized, then λ is C^r in p and thus, since $V_x = \lambda \circ \sigma^{-1}$ on R, it follows that V_x is still r-times continuously differentiable in x. In particular, and observing that we need to take a transpose in λ to keep the notation consistent, we get

$$V_{xx}(t, x(t, p))\frac{\partial x}{\partial p}(t, p) = \frac{\partial \lambda^T}{\partial p}(t, p)$$

which implies Eq. (2.32). $\square$

Note that, although it is not required that the parametrization of the trajectories is injective on the terminal manifold N, the function V always has a continuous extension to N (in the sense that the extended function is continuous on the domain $\overline{R} = R \cup N$). The reason is that $C(t_f(p), p) = \varphi(t_f(p), \xi(p))$, i.e., the cost $C(t, p)$ defined in (2.16) for points $t = t_f(p)$ only depends on the terminal point $\sigma(t_f(p), p) = (t_f(p), \xi(p))$ but not on the parameter p. Hence for $(t, x) \in N$ we can extend the definition of $V = C \circ \sigma^{-1}$ by taking any of the pre-images of σ

which are mapped to $\mathcal{T}_f = \{(t,p):\ t = t_f(p)\}$ by σ^{-1}. If the map σ extends to a diffeomorphism into a neighborhood of the section $\mathcal{T}_f$ (the terminal manifold N is then necessarily of codimension 1), the function V extends with the same smoothness properties as in Theorem 2.7 to a neighborhood of N.

We now show that the value function V corresponding to a parametrized family $\mathcal{E}$ of broken extremal lifts remains continuously differentiable at a switching surface provided certain regularity conditions are met. We first describe the local situation.

Definition 2.8. Let W denote a sufficiently small neighborhood of a point (t_0, p_0) which lies on the parametrization $\mathcal{T} = \{(t,p):\ t = \tau(p),\ p \in P\}$ of a switching surface. Let $D_0^* = \{(t,p) \in W:\ t > \tau(p)\}$, $D_1^* = \{(t,p) \in W:\ t < \tau(p)\}$, and denote the restrictions of the corresponding trajectories by subscripts. Also let $R_0 = \sigma(D_0^*)$, $S = \sigma(\mathcal{T})$, $R_1 = \sigma(D_1^*)$, and set $R = R_0 \cup S \cup R_1$. We say a C^r-parametrized family of broken extremal lifts has a *regular switching* at $(t_0, x_0) = (t_0, x(t_0, p_0))$ if the controls u_1 and u_2 extend as $C^{0,r}$ functions onto W and if the corresponding extensions σ_0 and σ_1 are $C^{1,r}$-diffeomorphisms on W.

Note that this definition only requires that σ_0 and σ_1 are injective separately. This does not yet imply that the associated flow σ of trajectories,

$$\sigma : (t,p) \mapsto (t, x(t,p)) = \begin{cases} (t, x_0(t,p)) & \text{if } (t,p) \in D_0^*, \\ (t, x_1(t,p)) & \text{if } (t,p) \in D_1^* \cup \mathcal{T}, \end{cases} \tag{2.34}$$

is injective.

Definition 2.9. We say a C^r-parametrized family $\mathcal{E}$ of broken extremal lifts has a *regular crossing* at $(t_0, x_0) = (t_0, x(t_0, p_0))$ if it has a regular switching at (t_0, x_0) and if the associated flow σ of trajectories is injective on a sufficiently small neighborhood W of (t_0, p_0).

If $(t_0, x_0) = (t_0, x(t_0, p_0))$ is a regular switching, then it follows that the switching surface $S = \sigma(\mathcal{T})$ is an imbedded submanifold of codimension 1. The associated flow σ will be injective, for instance, if the corresponding flows σ_0 and σ_1 are transversal to S and the corresponding dynamics point to the same sides of the tangent space to S at (t_0, x_0). These relations will be developed further in Section 3.

Definition 2.10. We say $\mathcal{E}$ defines a classical C^r-*field* of extremals over $D^* = \{(t,p) \in D:\ t_{\text{in}}(p) < t < t_f(p),\ p \in P\}$ if the map $\sigma : D^* \to \mathbb{R} \times \mathbb{R}^n$, $(t,p) \mapsto (t, x(t,p))$, is a $C^{1,r}$-diffeomorphism, i.e., is injective with a nonvanishing Jacobian determinant and is r-times continuously differentiable in p.

Definition 2.11. We say $\mathcal{E}$ defines a C^r-parametrized *field of broken extremals* over D_s^* if it is a C^r-parametrized family of broken extremal lifts for which the map σ is injective and the restriction σ_j of σ to $D_j^* = \{(t,p)\colon\ p \in P,\ t_{j+1}(p) < t < t_j(p)\}$ is a $C^{1,r}$-diffeomorphism for each $j = 0,\dots,m$.

Then we have the following result:

Theorem 2.12. *Let $\mathcal{E}$ define a C^1-parametrized field of normal broken extremals which has a regular crossing at $(t_0,x_0) = (t_0, x(t_0,p_0))$. Let W be a sufficiently small neighborhood of (t_0,p_0) such that the combined flow σ is injective and the individual flows σ_0 and σ_1 are regular on W. Then the associated value function $V: R = \sigma(W) \to \mathbb{R}$, $V = C \circ \sigma^{-1}$, is a continuously differentiable solution to the Hamilton–Jacobi–Bellman equation at (t_0,x_0).*

Proof. Since $\mathcal{E}$ has a regular switching at $(t_0,x_0) = (t_0, x(t_0,p_0))$ there exist $C^{0,1}$-extensions u_0 and u_1 of the controls to W and without loss of generality we may assume that the flow maps σ_i, $i = 1,2$, are diffeomorphisms on W. Let C_0 and C_1 be the corresponding extensions of the cost functions, i.e., if $C(\tau(p),p)$ denotes the cost of the parametrized family of extremals on the switching surface, then for $i = 0,1$ and all $(t,p) \in W$ we have

$$C_i(t,p) = \int_t^{\tau(p)} L(s, x_i(s,p), u_i(s,p))\,ds + C(\tau(p),p). \tag{2.35}$$

Then the functions $V_i: R \to \mathbb{R}$, $V_i = C_i \circ \sigma_i^{-1}$, are well defined and continuously differentiable on R. However, since V_i only satisfies the maximum condition on D_i^*, the functions are solutions to the Hamilton–Jacobi–Bellman equation only on $R_i = \sigma(D_i^*)$. For V_1 this follows since the Shadow-Price Lemma remains valid for broken extremals. The identities

$$\frac{\partial V_i}{\partial t}(t, x_i(t,p)) = -H(t, \lambda(t,p), x_i(t,p), u_i(t,p)), \tag{2.36}$$
$$\frac{\partial V_i}{\partial p}(t, x_i(t,p)) = \lambda(t,p) \tag{2.37}$$

remain valid on D_0^* and D_1^*, respectively. Hence the gradients of V_0 and V_1 are equal on N. (The Hamiltonian remains continuous at the switchings.) Thus, V_0 and V_1 are continuously differentiable functions on R and both functions and their gradients have identical values on S. Hence the composite function $V: R \to \mathbb{R}$ defined by

$$V(t,x) = \begin{cases} V_1(t,x) & \text{for } (t,x) \in R_1 \cup S, \\ V_0(t,x) & \text{for } (t,x) \in R_0 \end{cases} \tag{2.38}$$

is continuously differentiable on S with $\nabla V = \nabla V_0 = \nabla V_1$ and it solves the Hamilton–Jacobi–Bellman equation since V_0 and V_1 solve it on R_0 and R_1, respectively. □

Corollary 2.13. *Let $\mathcal{E}$ define a C^1-parametrized classical field or a field of normal broken extremals with regular crossings over D^*. Set $R = \sigma(D^*)$, $\overline{R} = \sigma(D^*) \cup N$ and let $V : R \to \mathbb{R}$, $V = C \circ \sigma^{-1}$, be the corresponding value function. Then V is a continuously differentiable solution to the Hamilton–Jacobi–Bellman equation on R which has a continuous extension to the terminal manifold N.*

Proof. For the case of a classical field of extremals (the flow is a diffeomorphism off the terminal manifold N) nothing needs to be shown. The result follows from Theorem 2.7. These arguments carry over to fields of broken extremals: Theorem 2.12 implies that V is a C^1-solution to the Hamilton–Jacobi–Bellman equation away from N. □

Corollary 2.13 gives the following result on optimality. Its proof is a classical and elementary argument which is omitted.

Corollary 2.14 [5,22]. *Let $\mathcal{E}$ be a C^r-parametrized field of normal broken extremals over D^* with regular crossings. Set $R = \sigma(D^*)$, $\overline{R} = \sigma(D^*) \cup N$, and let $V : R \to \mathbb{R}$, $V = C \circ \sigma^{-1}$, with continuous extension to $\overline{R}$ which satisfies $V(t, x) = \varphi(t, x)$ for all $(t, x) \in N$. Then the feedback control $u^* : R \to U$, $u^* = u \circ \sigma^{-1}$, is optimal on $\overline{R}$ (i.e., with respect to any other control for which the corresponding trajectory lies in R up to its terminal point in N) and the corresponding value function is given by V. In particular, V is continuously differentiable on R.*

3. Sufficient conditions for relative minima

In this section we derive sufficient conditions for an extremal $\Gamma_0 = (x(\cdot, p_0), u(\cdot, p_0))$ of a parametrized family of extremals to be locally optimal relative to other trajectories which stay close to $x(\cdot, p_0)$.

Definition 3.1. We say that an extremal Γ_0 defined over a compact interval $[t_{\mathrm{in}}(p_0), t_f(p_0)]$ provides a relative minimum over a set R if the restriction of the trajectory to $(t_{\mathrm{in}}(p_0), t_f(p_0))$ is contained in the interior of R and if any other trajectory which steers $x(t_{\mathrm{in}}(p_0), p_0)$ into the terminal manifold and lies in R does not give a better value for the cost.

Thus, the extremal Γ_0 is optimal over all trajectories which lie in R. It is not required that the corresponding controls remain close as well.

We assume as given a C^r-parametrized family of normal (broken) extremal lifts and give sufficient conditions under which this family locally defines a C^r-parametrized *field* of normal broken extremals *with regular crossings* near a reference parameter p_0. These conditions relate to the regularity of the map σ along the segments on which the broken extremals are smooth as well as to transversality conditions on the switching surfaces. We also need a local injectivity condition at the terminal manifold. In Section 3.2 we then give sufficient conditions for the required regularity of $\partial x/\partial p$ including transversality conditions at switchings.

3.1. Local imbeddings

Let $\mathcal{E}$ be a C^r-parametrized family of normal broken extremals with the parametrizations of the switching times given by continuously differentiable functions t_i, $i = 0, \dots, m+1$, $t_{\mathrm{in}} = t_{m+1} < t_m < \cdots < t_1 < t_0 = t_f$. In order to keep technical details at a minimum, we only consider the case when the terminal manifold is of codimension 1. We refer the reader to [18] for the considerably more technical general case. Henceforth we assume also that

(A) the control $u = u(t, p)$ extends as a $C^{0,1}$-function into a neighborhood D_f of the set $\mathcal{T}_f = \{(t, p) \in D\colon\ t = t_f(p)\}$ parametrizing the endpoints of trajectories in the terminal manifold N.

Under this assumption the system and adjoint equations can be defined also for $(t, p) \in D_f$ and without loss of generality we may assume the solutions exist on D_f. Note, however, that these extensions will not satisfy the minimum condition of the Maximum Principle for times $t > t_f(p)$.

Proposition 3.2. *Let $\mathcal{E}$ be a C^r-parametrized family of normal extremal lifts with codimension* 1 *terminal manifold N and suppose there exists a parameter $p_0 \in P$ such that $(\partial x/\partial p)(t, p_0)$ is nonsingular for $t_{\mathrm{in}}(p_0) \leqslant t \leqslant t_f(p_0)$. Then there exists a neighborhood W of p_0 so that the restriction of $\mathcal{E}$ to $\{(t, p)\colon\ t_{\mathrm{in}}(p) \leqslant t \leqslant t_f(p),\ p \in W\}$ defines a C^r-parametrized field of extremals.*

Proof. In this proposition we do not yet consider additional switchings off N and therefore $\sigma(t, p)$ is $C^{1,r}$. By the regularity of σ it follows from the implicit function theorem that for each point $\alpha \in [t_{\mathrm{in}}(p_0), t_f(p_0)] \times \{p_0\}$ there exists a neighborhood, G_α, of α on which σ is a $C^{1,r}$-diffeomorphism. Without loss of generality we may take G_α of the form $G_\alpha = I_\alpha \times W_\alpha$, where I_α is an open interval and W_α an open neighborhood of p_0. The sets $\{G_\alpha\colon\ \alpha \in [t_{\mathrm{in}}(p_0), t_f(p_0)] \times \{p_0\}\}$ form an open cover of $[t_{\mathrm{in}}(p_0), t_f(p_0)] \times \{p_0\}$ and by the Heine–Borel theorem there exists a finite subcover $\{G_{\alpha_i}\colon\ \alpha_i \in [t_{\mathrm{in}}(p_0), t_f(p_0)] \times \{p_0\}, i = 1, \dots, r\}$. Let $W = \bigcap_{i=1}^r W_{\alpha_i}$. Then the map $\sigma : D \to \mathbb{R} \times \mathbb{R}^n$ restricted to $D^* = \{(t, p)\colon\ t_{\mathrm{in}}(p) < t < t_f(p),\ p \in W\}$ is a $C^{1,r}$-diffeomorphism. For, if $\sigma(t_1, p_1) = \sigma(t_2, p_2)$, then trivially $t_1 = t_2$ and this time lies in some interval I_{α_i}. Without loss of generality say $t_1 \in I_{\alpha_i}$. But σ is a $C^{1,r}$-diffeomorphism on $I_{\alpha_i} \times U_{\alpha_i}$ and since both p_1 and p_2 lie in $W \subseteq W_{\alpha_i}$, we must have $p_1 = p_2$ as well. Thus σ is injective on D^*. This proves the theorem. $\square$

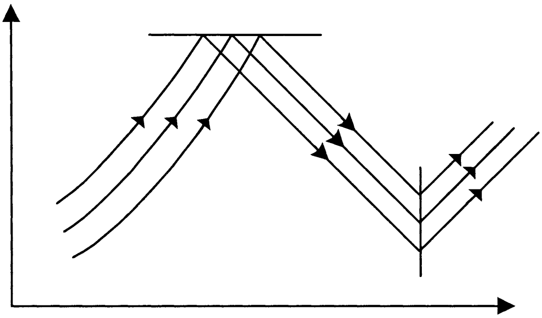

Fig. 1. Intersections of a flow of broken extremals.

We extend this construction to families of normal broken extremals. It is clear from Theorem 3.2 that the restrictions of the flow to the sets $D_i = \{(t, p)\colon t_{i+1}(p) \leqslant t \leqslant t_i(p),\ p \in W\}$ will be a field of smooth extremals provided the matrix $(\partial x/\partial p)(t, p_0)$ is nonsingular on $[t_{i+1}(p_0), t_i(p_0)]$ and W is chosen as a sufficiently small neighborhood of p_0. However, this condition for each of the subintervals is not sufficient to guarantee that we can imbed the reference trajectory $x(t, p_0)$, $t_{\text{in}}(p_0) \leqslant t \leqslant t_f(p_0)$, into a field of broken extremals, as the example in Fig. 1 shows. Clearly, each of the restricted flows defines a field, but the composite flow does not because of the overlap near the switching surface. This is the typical conjugate point behavior [26] as can also be seen near fold-singularities for smooth families of parametrized extremals (see [27]). This behavior needs to be excluded by appropriate transversality conditions.

Let $\mathcal{E}$ be a C^r-parametrized family of normal broken extremal lifts and consider the parametrization $\mathcal{T} = \{(t, p)\colon t = \tau(p),\ p \in W\}$ of a switching surface S near p_0, $S = \{(t, x)\colon t = \tau(p),\ x = x(\tau(p), p),\ p \in W\}$. Assume there exists a continuously differentiable function $\psi = \psi(t, x)$ so that $S = \{(t, x)\colon \psi(t, x) = 0\}$. Then the flow of the parametrized family of extremals crosses S transversally if for all $(t, x) \in S$

$$\psi_t(t, x) + \psi_x(t, x) f(t, x, u_i) > 0, \quad i = 1, 2, \tag{3.1}$$

where u_1 and u_2 denote the controls prior to and after the switching. The positive sign is chosen without loss of generality.

Definition 3.3. We say a C^r-parametrized family $\mathcal{E}$ of normal broken extremal lifts has regular and transversal crossings at p_0 if all switching surfaces $S_i = \{(t, x)\colon t = t_i(p),\ x = x(t_i(p), p),\ p \in W\}$ for $i = 1, \ldots, m$ are imbedded codimension 1 submanifolds and if the flow of extremals has regular crossings and is transversal to the switching surfaces S_i at $(t_i, x_i) = (t_i(p_0), x(t_i(p_0), p_0))$.

Theorem 3.4. *Let $\mathcal{E}$ be a C^r-parametrized family of normal broken extremal lifts with codimension* 1 *terminal manifold N and suppose there exists a $p_0 \in P$*

such that (i) *the matrix* $(\partial x_i/\partial p)(t, p_0)$ *is nonsingular on* $t_{i+1}(p_0) \leqslant t \leqslant t_i(p_0)$ *for* $i = 0, \dots, m$, *and* (ii) *the trajectory* $x(t, p_0)$ *has regular and transversal crossings. Then there exists a neighborhood* W *of* p_0 *such that the restriction of* $\mathcal{E}$ *to* $\{(t, p): t_{\text{in}}(p) \leqslant t \leqslant t_f(p),\ p \in W\}$ *defines a* C^r*-parametrized field of normal broken extremals which has regular and transversal crossings.*

Proof. Without loss of generality we consider extremals with only 1 switching surface. We can use the arguments in the proof of Theorem 3.2 to show that there exist neighborhoods W_0 and W_1 of p_0, such that the restrictions of $\mathcal{E}$ to $\{(t, p): t_{\text{in}}(p) \leqslant t \leqslant t_1(p),\ p \in W_1\}$ and $\{(t, p): t_1(p) \leqslant t \leqslant t_f(p),\ p \in W_0\}$ define C^r-fields of extremals. We now use the transversal crossings of the extremals to show that there exists a $W \subset W_0 \cap W_1$ so that the map σ is injective on $D = \{(t, p): t_{\text{in}}(p) \leqslant t \leqslant t_f(p), p \in W\}$ and thus that the restriction of $\mathcal{E}$ to D defines a C^r-field of broken extremals.

Since $(\partial x_0/\partial p)(t_1(p_0), p_0)$ is nonsingular by assumption, the map $\sigma_0 : (t, p) \mapsto (t, x_0(t, p))$ (defined via the $C^{0,r}$-extension of $u(t, p)$) is a $C^{1,r}$-diffeomorphism near (t_0, p_0) with inverse $\sigma_0^{-1} : (t, x) \mapsto (t, \pi(t, x))$. Hence near $(t_0, x_0) = (t_0, x(t_0, p_0))$ the switching surface S can be described as $S = \{(t, x): \psi(t, x) = t - \tau(\pi(t, x)) = (\mathrm{Id}, -\tau) \circ \sigma_0^{-1}(t, x) = 0\}$ and $\nabla\psi(t_0, x_0) \neq 0$ since

$$\begin{aligned}
\nabla\psi(t, x) &= \big(1,\ -\nabla\tau(p)\big)\begin{pmatrix} 1 & 0 \\ \frac{\partial x_0}{\partial t}(\tau(p), p) & \frac{\partial x_0}{\partial p}(\tau(p), p)\end{pmatrix}^{-1} \\
&= \big(1,\ -\nabla\tau(p)\big)\begin{pmatrix} 1 & 0 \\ -\big(\frac{\partial x_0}{\partial p}\big)^{-1}\frac{\partial x_0}{\partial t} & \big(\frac{\partial x_0}{\partial p}\big)^{-1}\end{pmatrix} \\
&= \Big(1 + \nabla\tau(p)\big(\tfrac{\partial x_0}{\partial p}(\tau(p), p)\big)^{-1} f(\tau(p), x(p), u_0(p)), \\
&\qquad -\nabla\tau(p)\big(\tfrac{\partial x_0}{\partial p}(\tau(p), p)\big)^{-1}\Big).
\end{aligned}$$

Note that

$$\frac{\partial\psi}{\partial t}(t_0, x_0) + \frac{\partial\psi}{\partial x}(t_0, x_0) f(t_0, x_0, u_0(t_0, p_0)) \equiv 1. \tag{3.2}$$

Thus, S is an imbedded n-dimensional submanifold. The transversality condition (3.1) on ψ implies that

$$\frac{d\psi}{dt}\big(t_1(p_0), x_i(t_1(p_0), p_0)\big) > 0, \quad i = 0, 1.$$

It follows that there exists an $\epsilon > 0$ such that the above inequality will hold for (t, p) satisfying $\|t - t_1(p_0)\| < \epsilon$ and $\|p - p_0\| < \epsilon$. By the continuity of $t_1(\cdot)$ there exists an open neighborhood $\tilde{W} \subset \{p: \|p - p_0\| < \epsilon\}$ so that $t_1(\tilde{W}) \subset (t_1(p_0) - \epsilon, t_1(p_0) + \epsilon)$. We choose a $\delta > 0$ small enough so that $W = \{p: \|p - p_0\| < \delta\} \subset W_0 \cap W_1 \cap \tilde{W}$. Injectivity of the flow will follow if we can show that $S = Y_0 \cap Y_1$, where $Y_i = \sigma(D_i)$, $i = 0, 1$, and $D_0 = \{(t, p): t_1(p) \leqslant t \leqslant t_f(p),\ p \in W\}$, $D_1 = \{(t, p): t_{\text{in}}(p) \leqslant t \leqslant t_1(p),\ p \in W\}$. It is apparent that

$S \subset Y_0 \cap Y_1$. We now prove that $Y_0 \cap Y_1 \subset S$, or equivalently, $S^c \subset (Y_0 \cap Y_1)^c$, where the superscript "c" denotes the set complement. Let $(t,x) \in S^c$ and suppose $(t,x) \in Y_0 \cap Y_1$. It follows that there exist $(t,p_1) \in D_0$ and $(t,p_2) \in D_1$ such that $\sigma_0(t,p_1) = (t,x) = \sigma_1(t,p_2)$. Since $(t,x) \in S^c$ we have $t_1(p_1) < t < t_1(p_2)$ and thus $p_1 \neq p_2$. Evaluating ψ at the switching surface we get

$$\psi\big(t_1(p_1), x_0(t_1(p_1), p_1)\big) = \psi\big(t_1(p_2), x_1(t_1(p_2), p_2)\big) = 0.$$

By construction the functions $\psi(t, x_i(t, p_{i+1}))$, $i = 0, 1$, are monotonically increasing on $[t_1(p_1), t_1(p_2)]$ and thus

$$\begin{aligned}\psi(t,x) &= \psi(t, x_0(t,p_1)) > \psi\big(t_1(p_1), x_0(t_1(p_1), p_1)\big) = 0\\ &= \psi\big(t_1(p_2), x_1(t_1(p_2), p_2)\big) > \psi(t, x_1(t,p_2)) = \psi(t,x).\end{aligned}$$

Contradiction. This proves that the combined flow is injective near the switching surface; i.e., we have a local field. □

Corollary 3.5. *Let $\mathcal{E}$ be the C^r-parametrized family of normal broken extremal lifts with codimension* 1 *terminal manifold N satisfying conditions* (i) *and* (ii) *of Theorem* 3.4 *and let W be a neighborhood of p_0 for which $\mathcal{E}$ is a field with regular and transversal crossings. Then setting $R = \sigma(D^*)$ and $\overline{R} = \sigma(D^*) \cup N$, the mapping $V : \overline{R} \to \mathbb{R}$, $V = C \circ \sigma^{-1}$, is a continuously differentiable solution to the Hamilton–Jacobi–Bellman equation on R which has a continuous extension onto $\overline{R}$ and the extremal $\Gamma_0 = (x(\cdot, p_0), u(\cdot, p_0))$ is a relative minimum.*

3.2. Sufficient conditions for regular and transversal crossings

We now give criteria which allow to verify conditions (i) and (ii) of Theorem 3.4 (i.e., conditions for the flow to be regular and to have transversal crossings). We start with characterizing when the map σ is regular. For this we simply modify classical results on Riccati equations to our set-up (see also [27]). This requires that the adjoint variables are differentiable in p and thus we consider only *nicely parametrized families of extremal lifts*.

Corollary 3.6. *Let $\mathcal{E}$ be a nicely C^1-parametrized family of normal extremal lifts and suppose the map $\sigma : D \to \mathbb{R}^n$, $(t,p) \mapsto x(t,p)$, is a C^1-diffeomorphism of some open subset $O \subset \operatorname{int} D$ onto an open subset $R \subset \mathbb{R}^n$. Then for t such that $(t,p) \in O$, the function*

$$S(t,p) = V_{xx}(t, x(t,p)) = \frac{\partial \lambda^T}{\partial p}(t,p)\left(\frac{\partial x}{\partial p}(t,p)\right)^{-1} \tag{3.3}$$

satisfies the differential equation

$$\dot S + S f_x + f_x^T S + H_{xx} + (S f_u + H_{xu}) \frac{\partial u}{\partial p}\left(\frac{\partial x}{\partial p}\right)^{-1} \equiv 0, \tag{3.4}$$

where the partial derivatives of f and H are evaluated along the extremal corresponding to the parameter p.

Proof. The matrices $\partial x/\partial p$ and $\partial\lambda^T/\partial p$ satisfy the variational equations of, respectively, the dynamics and the covariational equations. Equation (3.4) follows by a direct calculation from these differential equations:

$$\begin{aligned}
\dot S &= \left[\frac{d}{dt}\left(\frac{\partial\lambda^T}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} + \left(\frac{\partial\lambda^T}{\partial p}\right)\left[\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right)^{-1}\right] \\
&= \left[\frac{d}{dt}\left(\frac{\partial\lambda^T}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} - \left(\frac{\partial\lambda^T}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1}\left[\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} \\
&= \left(-\left(L_{xx}\frac{\partial x}{\partial p} + L_{xu}\frac{\partial u}{\partial p}\right) - f_x^T\frac{\partial\lambda^T}{\partial p} - \lambda\left(f_{xx}\frac{\partial x}{\partial p} + f_{xu}\frac{\partial u}{\partial p}\right)\right)\left(\frac{\partial x}{\partial p}\right)^{-1} \\
&\quad - S\left(f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1} \\
&= -Sf_x - f_x^T S - H_{xx} - (Sf_u + H_{xu})\left(\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1}. \qquad \square
\end{aligned}$$

We will now use Eq. (3.4) to give equivalent characterizations of the regularity of the map σ. For the case when the control takes values in the interior of the control set, these are classical results about linear systems and Riccati equations which we briefly recall (see, e.g., [28]). Then we will give conditions for the regularity of broken extremals.

Suppose the control u takes values in the interior of the control set on an interval I or, more generally, assume H_u vanishes identically along (λ, x, u) along I. In this case it follows from the minimum condition that the Legendre condition holds in the sense that H_{uu} is positive semidefinite along (λ, x, u). If H_{uu} is positive definite we say that the *strengthened Legendre condition* holds. Then

$$\frac{\partial u}{\partial p} = -H_{uu}^{-1}\left(H_{ux}\frac{\partial x}{\partial p} + f_u^T\frac{\partial\lambda^T}{\partial p}\right) \tag{3.5}$$

and under the assumptions of Corollary 3.6 we can eliminate the control term from Eq. (3.4) to get the customary Riccati equation for the second derivatives V_{xx}:

$$\dot S = -Sf_x - f_x^T S - H_{xx} + (Sf_u + H_{xu})H_{uu}^{-1}\left(H_{ux} + f_u^T S\right).$$

It is well known that *the existence of a bounded solution to this Riccati equation over an interval I is equivalent to the regularity of the Jacobian $\partial x/\partial p$* [28]. This follows from the lemma below which goes back to Legendre and the calculus of variations.

Lemma 3.7 [29]. *Let A, B, M, N, X_0 and Y_0 be $(n \times n)$-matrices defined on an open interval I containing t_{in} and assume X_0 is nonsingular. Let $(X(t), Y(t))$ be a solution to the initial value problem*

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} A(t) & -M(t) \\ -N(t) & -B(t) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$
$$\begin{pmatrix} X(t_{\mathrm{in}}) \\ Y(t_{\mathrm{in}}) \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}. \tag{3.6}$$

The matrix X is nonsingular on I if and only if there exists a solution S on I for the Riccati equation

$$\dot{S} + SA(t) + B(t)S - SM(t)S + N(t) \equiv 0,$$
$$S(t_{\mathrm{in}}) = S_0 = Y_0 X_0^{-1}. \tag{3.7}$$

From Lemma 3.7 we obtain the following characterization for the regularity of σ:

Corollary 3.8. *Let $\mathcal{E}$ be a nicely C^1-parametrized family of normal extremal lifts and let $\Gamma_p : [t_{\mathrm{in}}(p), t_f(p)] \to \mathbb{R}^n \times U$ be an extremal which satisfies $H_u = 0$ and the strengthened Legendre condition $H_{uu} > 0$. Suppose $(\partial x/\partial p)(t_f(p), p)$ is nonsingular. Then $(\partial x/\partial p)(t, p)$ is nonsingular on $[\tau, t_f(p)]$, $\tau \geqslant t_{\mathrm{in}}(p)$, if and only if the solution $S(t, p)$ to the Riccati equation* (3.8),

$$\dot{S} + Sf_x + f_x^T S + H_{xx} - (Sf_u + H_{xu}) H_{uu}^{-1} \left(H_{ux} + f_u^T S \right) \equiv 0, \tag{3.8}$$

with terminal condition

$$S(t_f(p), p) = \frac{\partial \lambda^T}{\partial p}(t_f(p), p) \left(\frac{\partial x}{\partial p}(t_f(p), p) \right)^{-1}$$

exists on $[\tau, t_f(p)]$.

If the control u does not depend on the parameter p, then (3.4) simplifies to a linear Lyapunov differential equation,

$$\dot{S} + Sf_x + f_x^T S + H_{xx} \equiv 0. \tag{3.9}$$

But this equation always has a solution under our assumptions. Therefore we obtain the following result from Lemma 3.7:

Corollary 3.9. *Let $\mathcal{E}$ be a nicely C^1-parametrized family of normal extremal lifts and let $\Gamma_p : [t_{\mathrm{in}}(p), t_f(p)] \to \mathbb{R}^n \times U$ be an extremal for which the corresponding control does not depend on p; i.e., $(\partial u/\partial p)(t, p) \equiv 0$. Then $(\partial x/\partial p)(t, p)$ is nonsingular on $[t_{\mathrm{in}}(p), t_f(p)]$ if $(\partial x/\partial p)(t_f(p), p)$ is nonsingular.*

This situation arises in the analysis of bang–bang trajectories when consequently the regularity of the map σ only depends on the behavior at the switching surfaces. We now analyze the behavior of the flow around a switching surface. Suppose the switching surface is described in the parameter space by a C^r-function $\tau : P \to \mathbb{R}$, $p \mapsto \tau(p)$, and let $\mathcal{T} = \{(t, p)\colon\ p \in P,\ t = \tau(p)\}$. The switching surface $S = \{(t, x)\colon\ t = \tau(p),\ x = x(\tau(p), p),\ p \in P\}$ will be an n-dimensional imbedded submanifold near $(t_0, x_0) = (\tau(p_0), x(\tau(p_0), p_0))$ if the map $\Xi : P \to S$, $p \mapsto (\tau(p), x(\tau(p), p))$, has a nonsingular Jacobian at p_0. Also let $D_0 = \{(t, p)\colon\ \tau(p) \leqslant t \leqslant t_f(p),\ \ p \in P\}$, $D_1 = \{(t, p)\colon\ t_{\text{in}}(p) \leqslant t \leqslant \tau(p),$ $p \in P\}$, and denote the controls and trajectories on D_i by a subscript $i = 0, 1$. We *assume* that the controls $u_i = u_i(t, p)$ extend as $C^{0,r}$-functions into open neighborhoods $\tilde{D}_i$ of D_i, $i = 0, 1$ (i.e., extend beyond the switching surface $\mathcal{T}$). Thus the flows $x_i(t, p)$ and the adjoint variables $\lambda_i(t, p)$ also extend as $C^{1,r}$-functions into a neighborhood of D_i and without loss of generality we can take it as $\tilde{D}_i$, $i = 0, 1$. Like in the proof of Lemma 2.5 the following relation follows from Lemma 2.6:

$$\begin{aligned}
&\frac{\partial x_1}{\partial p}(\tau(p), p) - \frac{\partial x_0}{\partial p}(\tau(p), p)\\
&\quad = \big[f\big(\tau(p), x_0(\tau(p), p), u_0(\tau(p), p)\big)\\
&\qquad - f\big(\tau(p), x_1(\tau(p), p), u_1(\tau(p), p)\big)\big]\nabla\tau(p),
\end{aligned} \tag{3.10}$$

where the gradient is a row vector. Note that $(\partial x_0/\partial p)(\tau(p), p)$ and $(\partial x_1/\partial p) \times (\tau(p), p)$ differ by a rank 1 matrix. The following relation from linear algebra is well known [30]:

Lemma 3.10. *Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular and $v \in \mathbb{R}^n$ is a nonzero vector. Then $B = A + uv^T$ is nonsingular if and only if $1 + v^T A^{-1}u \neq 0$. In this case*

$$\big(A + uv^T\big)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}.$$

To simplify the notation we set $x(p) = x_0(\tau(p), p) = x_1(\tau(p), p)$, $u_0(p) = u_0(\tau(p), p)$, and $u_1(p) = u_1(\tau(p), p)$. Then we have

Corollary 3.11. *Suppose the controls u_0 and u_1 extend as $C^{0,r}$-functions into open neighborhoods of the set $\{(t, p)\colon\ t = \tau(p)\}$. If $(\partial x_0/\partial p)(t, p)$ is nonsingular for $t = \tau(p)$, then $(\partial x_1/\partial p)(t, p)$ is nonsingular for $t = \tau(p)$ if and only if*

$$\begin{aligned}
&\nabla\tau(p)\left(\frac{\partial x_0}{\partial p}(\tau(p), p)\right)^{-1}\\
&\quad \times \big(f(\tau(p), x(p), u_0(p)) - f(\tau(p), x(p), u_1(p))\big) \neq -1.
\end{aligned} \tag{3.11}$$

Proof. The statement is trivially correct if $\nabla\tau(p)=0$ and follows by applying Lemma 3.10 to (3.10) otherwise. □

Condition (3.11) is equivalent to a transversality condition on the flow of x_1 at the switching surface. For, if $(\partial x_0/\partial p)(t,p)$ is nonsingular at $t=\tau(p)$, then the map $\sigma_0:(t,p)\mapsto(t,x_0(t,p))$ is locally a $C^{1,r}$-diffeomorphism and hence invertible. As before we define the switching surface $S=\{(t,x)\colon t=\tau(p),\ x=x(p)=x_0(\tau(p),p)\}$, in the (t,x)-space in a neighborhood W of the reference point $(t_p,x_p)=(\tau(p),x(p))$ as $S=\{(t,x)\in W\colon \psi(t,x)=0\}$, where $\psi:W\to\mathbb{R}$ is given by $\psi(t,x)=(I,-\tau)\circ\sigma_0^{-1}(t,x)$ and is $C^{1,r}$. Differentiating ψ gives

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}(\tau(p),x(p))+\frac{\partial\psi}{\partial x}(\tau(p),x(p))f(\tau(p),x(p),u_1(p))\\
&\quad=1+\nabla\tau(p)\left(\frac{\partial x_0}{\partial p}(\tau(p),p)\right)^{-1}\\
&\qquad\times\big(f(\tau(p),x(p),u_0(p))-f(\tau(p),x(p),u_1(p))\big).
\end{aligned}\tag{3.12}$$

Furthermore, by construction (see also Theorem 3.4)

$$\frac{\partial\psi}{\partial t}(\tau(p),x(p))+\frac{\partial\psi}{\partial x}(\tau(p),x(p))f(\tau(p),x(p),u_0(p))\equiv 1.\tag{3.13}$$

Thus, we have

Theorem 3.12. *Suppose the controls u_0 and u_1 extend as $C^{0,r}$-functions onto a neighborhood of $\{(t,p)\colon t=\tau(p)\}$ and suppose $(\partial x_0/\partial p)(t,p)$ is nonsingular at $t=\tau(p_0)$. Then the switching surface $S=\{(t,x)\colon t=\tau(p),\ x=x(\tau(p),p),\ p\in P\}$ is an n-dimensional imbedded submanifold near $(t_0,x_0)=(\tau(p_0),x(\tau(p_0),p_0))$. The combined flow $\sigma:(t,p)\mapsto(t,x(t,p))$ has regular and transversal crossings at the switching surface S at (t_0,x_0) if and only if*

$$\begin{aligned}
&1+\nabla\tau(p_0)\left(\frac{\partial x_0}{\partial p}(\tau(p_0),p_0)\right)^{-1}\\
&\quad\times\big(f(t_0,x_0,u_0(p_0))-f(t_0,x_0,u_1(p_0))\big)>0.
\end{aligned}\tag{3.14}$$

This theorem characterizes transversal crossings in the parameter space. Typically this is how the data in a regular synthesis will be constructed. Furthermore, under this condition the regularity of σ is guaranteed for the flow of σ_1 at the new terminal manifold. This then allows us to characterize the regularity of σ on $[t_{\mathrm{in}}(p),\tau(p)]$ using the earlier results with $(\partial x_1/\partial p)(\tau(p),p)$ given by (3.10) as terminal condition.

In order to apply Theorem 3.12 we still need an effective way to calculate (3.14). Such a procedure indeed exists whenever the “switchings” occur on surfaces which are zero sets of smooth functions in (t, p)-space. For example, in the case of bang–bang trajectories typically the parametrizations $t = \tau(p)$ of the switching surfaces are obtained by solving an equation of the type $\Phi(t, p) = 0$ given by some switching function for t. This indeed allows to calculate the quantity

$$\nabla\tau(p)\left(\frac{\partial x_0}{\partial p}(\tau(p), p)\right)^{-1} \tag{3.15}$$

without calculating any partial derivatives with respect to p or matrix inversions. However, the specifics depend on the special form of the dynamics. Here we only develop the equations for the optimal control problem Σ_1 considered in Section 2.2. Recall that the switching function is given by

$$\Phi(t, p) = 1 + \lambda(t, p)g(t, x(t, p)) \tag{3.16}$$

and the control in the parametrized family of extremals satisfies

$$u(t, p) = \begin{cases} 1 & \text{if } \Phi(t, p) < 0, \\ 0 & \text{if } \Phi(t, p) > 0. \end{cases} \tag{3.17}$$

Thus $t = \tau(p)$ is the local solution of the equation $\Phi(t, p) = 0$ near a switching (t_0, p_0) of the reference trajectory. Such a solution exists by the implicit function theorem if the time-derivative $\dot{\Phi}(t_0, p_0)$ does not vanish (see (2.14)). It follows by implicit differentiation that

$$\nabla\tau(p_0) = -\frac{(\partial\Phi/\partial p)(t_0, p_0)}{\dot{\Phi}(t_0, p_0)}. \tag{3.18}$$

Using $S(t, p) = \lambda(t, p)((\partial x/\partial p)(t, p))^{-1}$, which in the case of bang–bang trajectories is easily computed as solution to the linear equation (3.9), we have that

$$\begin{aligned} \frac{\partial\Phi}{\partial p}(\tau(p), p) &= \lambda(t, p)D_x g(t, x(t, p))\frac{\partial x}{\partial p}(t, p) + g^T(t, x(t, p))\frac{\partial\lambda^T}{\partial p}(t, p) \\ &= \left\{\lambda(t, p)D_x g(t, x(t, p)) + g^T(t, x(t, p))S(t, p)\right\}\frac{\partial x}{\partial p}(t, p). \end{aligned} \tag{3.19}$$

Setting $x_0 = x(t_0, p_0)$ we therefore obtain

$$\begin{aligned} &\nabla\tau(p_0)\left(\frac{\partial x_0}{\partial p}(t_0, p_0)\right)^{-1} \\ &\quad = -\frac{1}{\dot{\Phi}(t_0, p_0)}\left(\lambda(t_0, p_0)D_x g(t_0, x_0) + g^T(t_0, x_0)S(t_0, p_0)\right). \end{aligned}$$

Further simplifications in (3.14) can be made because of the special form of the dynamics. Let $\Delta u = u_0(t_0, p_0) - u_1(t_0, p_0)$ denote the jump in the control, at the switching surface. It follows from the minimization property of the controls that

$$\Delta u = -\operatorname{sgn}\dot{\Phi}(t_0, p_0) \tag{3.20}$$

and thus we have

$$1+\nabla\tau(p_0)\left(\frac{\partial x_0}{\partial p}(t_0,p_0)\right)^{-1}\bigl(f(t_0,x_0,u_0(p_0))-f(t_0,x_0,u_1(p_0))\bigr)$$
$$=1+\frac{1}{|\dot{\Phi}(t_0,p_0)|}\bigl(\lambda(t_0,p_0)D_xg(t_0,x_0)+g^T(t_0,x_0)S(t_0,p_0)\bigr)g(t_0,x_0). \tag{3.21}$$

Hence we have the following result:

Theorem 3.13. *Consider the optimal control problem Σ_1 and suppose the reference control $u(\cdot,p_0)$ has a bang–bang switch at t_0 with $\dot{\Phi}(t_0,p_0)\neq 0$. Then the switching surface S is an n-dimensional imbedded submanifold near (t_0,x_0), $x_0=x(t_0,p_0)$, and there exists a continuously differentiable function τ defined in some neighborhood W of p_0 such that $S=\{(t,x)\colon\ t=\tau(p),\ x=x(\tau(p),p),\ \ p\in W\}$. Assuming that $(\partial x_0/\partial p)(t_0,p_0)$ is nonsingular, the combined flow $\sigma:(t,p)\mapsto(t,x(t,p))$ has a regular and transversal crossing at S at (t_0,x_0) if and only if*

$$|\dot{\Phi}(t_0,p_0)|+\bigl\{\lambda(t_0,p_0)D_xg(t_0,x_0)+g^T(t_0,xp_0)S(t_0,p_0)\bigr\}g(t_0,x_0)>0. \tag{3.22}$$

Thus all necessary calculations are subsumed in the computation of S. Based on the formulas given in this paper, it is not difficult to develop an algorithmic scheme which verifies whether a given trajectory has transversal crossings by propagating the solution S to (3.4) between the switching surfaces. The precise formulas, however, depend on the model and we only refer the reader to [23,24] where this algorithm has been developed for bang–bang controls in a problem of cancer chemotherapy. If some of the controls are not constant, it is possible that the solution S need not exist over the full interval and this corresponds to conjugate points and singularities in the value-function. In the family of parametrized extremal lifts this shows in singularities in the parametrizations and we refer the reader to [27] where fold and cusp-singularities have been analyzed from this aspect.

4. Conclusion

Our results provide an effective method to determine the local optimality of a reference trajectory where the control is allowed to have discontinuities. It is not claimed that our result can be used to analyze all possible scenarios. Clearly this is not the case. In some sense we developed sufficient conditions for a local minimum in the so-called nondegenerate case. In our results we were always assuming the *most regular structure possible*. Trajectories were assumed to cross switching surfaces transversally. Tangential crossings were not allowed and indeed may lead to much more complicated behaviors. Similarly, switching surfaces were submanifolds parametrized locally as functions $\tau = \tau(p)$, not just arbitrary n-dimensional imbedded submanifolds of the (t, x)-space. There is no reason why one should be able to parametrize the switching times in p, but it certainly is the most regular behavior. For instance, in [23] and [24] mathematical models for optimal control of cancer chemotherapy are considered described by bilinear systems. In these models singular controls are not optimal and the local optimality of bang–bang trajectories can be established using our framework. More degenerate phenomena require extra and increasingly more complex and difficult analysis. It would be interesting to pursue some of these more degenerate cases to understand the limitations of the method of characteristics as it is developed here.

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